Machine Learning with Graphs: Spectral Graph Theory 2/3 - Laplacian Eigenvectors and Eigenvalues (cont.)

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The Laplacian matrix (cont.)

We will continue developing our understanding of the spectrum of the graph Laplacian using examples.

A *ring (or cycle) graph* has a Laplacian matrix as following ($n = 4$):

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Matrices with the structure above—rows have the same elements rotated to the right—are called *circulant* and are a special case of *Toeplitz matrices*. An eigenvector § of L must satisfy the following:

$$L\mathbf{x} = \begin{bmatrix} 2x_1 - x_2 - x_4 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_2 - x_4 \\ 2x_4 - x_2 - x_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Assuming that $x_{-1} = x_n$ and $x_{n+1} = 1$—i.e. indices are modulo n and shifted by 1—we can write $2x_i - x_{i-1} i x_{i+1} = \lambda x_i$. Thus, we know that if $\mathbf{x} = [x_1, x_2, \ldots x_n]^T$ is an eigenvector of L, the same holds for $[x_2, x_3, \ldots x_n, x_1]$ and any other cyclic permutation of $\mathbf{x}$. This gives us a hint on the how these eigenvectors should look like. Assuming that n is even, $\mathbf{x}_0, \ldots \mathbf{x}_{n/2}, \mathbf{y}_0, \ldots \mathbf{y}_{n/2-1}$ are eigenvectors of L, as defined below, are eigenvectors of L:

$$\mathbf{x}_k = \left[\cos(0), \cos\left(\frac{2\pi k}{n}\right), \cos\left(\frac{4\pi k}{n}\right), \ldots \cos\left(\frac{2(n-1)\pi k}{n}\right)\right]$$

for $k = 0, 1, \ldots n/2$

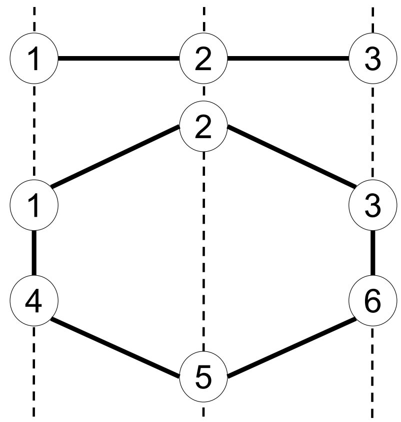

Figure 1: Projection of a path graph on a ring graph.

$$\mathbf{y}_k = \left[\sin(0), \sin\left(\frac{2\pi k}{n}\right), \sin\left(\frac{4\pi k}{n}\right), \ldots \sin\left(\frac{2(n-1)\pi k}{n}\right)\right]$$

for $k = 1, 2, \ldots (n/2) - 1$

Using the following identities:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

We can show that:

$$2\cos\left(\frac{2\pi kj}{n}\right) - \cos\left(\frac{2\pi k(j-1)}{n}\right) - \cos\left(\frac{2\pi k(j+1)}{n}\right) = \left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right)\cos\left(\frac{2\pi kj}{n}\right)$$

We can use a similar reasoning to show that $(2 - 2\cos((2\pi k)/n))$ is also an eigenvalue for $\mathbf{y}_k$.

Let's consider now the case of a *path graph*, which is a bit more complex than for rings graphs. Still, the spectrum of paths and rings are quite related. We will show that the eigenvectors of a path graph are as follows:

$$\mathbf{x}_k = \left[\cos\left(\frac{\pi k}{n} - \frac{\pi k}{2n}\right), \cos\left(\frac{2\pi k}{n} - \frac{\pi k}{2n}\right), \ldots \cos\left(\frac{n\pi k}{n} - \frac{\pi k}{2n}\right)\right]$$

for $k = 0, 1, \ldots n-1$

Moreover, the associated eigenvalues are in the form $2 - 2\cos((\pi k)/n))$.

To show this, we will use the a *projection trick* illustrated in Figure 1. A path with n vertices is projected on a ring with $2n$ vertices. More specifically, a vertex with index i is mapped to vertices with indices i and $n+i$ in the ring. We will show how the eigenvectors of these two graphs are related. Let L_P and L_R be the Laplacians of the path and ring, respectively. We can show the following:

$$[I_n, I_n]L_R \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 2L_P$$

where I_n is an $n \times n$ identity matrix.

Now, assume that we have an eigenvector $\mathbf{y} = [\mathbf{y}^{(1)}, \mathbf{y}^{(2)}]$ of L_R—i.e. $y_i = y_{n+i}, \forall i$. Then, we can show that $\mathbf{x} = \mathbf{y}^{(1)}$ is an eigenvector of L_P.

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathbf{x} = \mathbf{y}$$

$$L_R \begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathbf{x} = \lambda \mathbf{y}$$

$$[I_n, I_n]L_R \begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathbf{x} = 2L_P\mathbf{x} = 2\lambda\mathbf{x}$$

Now we know how to find $\mathbf{x}$ from $\mathbf{y}$, but how do we find $\mathbf{y}$? We can show that this holds for:

$$\mathbf{y}_k = \left[\cos\left(\frac{\pi k}{n} - \frac{\pi k}{2n}\right), \cos\left(\frac{2\pi k}{n} - \frac{\pi k}{2n}\right), \ldots \cos\left(\frac{2n\pi k}{n} - \frac{\pi k}{2n}\right)\right]$$

for $k = 0, 1, \ldots n-1$

First, we have to show that $[\mathbf{y}_k]_i = [\mathbf{y}_k]_{n+1}$:

$$\cos\left(\frac{\pi k j}{n} - \frac{\pi k}{2n}\right) = \cos\left(\frac{\pi k(n+j)}{n} - \frac{\pi k}{2n}\right) =$$

which can be done again using the same trigonometric identities used earlier.

We can also use the trigonometric identities to show that $\dagger_k$ is an eigenvector of L_R (ring graph), and thus $2[\mathbf{y}_k]_j - [\mathbf{y}_k]_{j-1} - [\mathbf{y}_k]_{j+1} = \lambda_k[\mathbf{y}_k]_j$. In fact, this holds for $\lambda_k = (2 - 2\cos((2\pi k)/n))$.

References

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[3] Daniel Spielman. Spectral graph theory. *Combinatorial scientific computing*, 2012.