# Logic B-Exam
May 1997

There are 4 questions worth 10 points each for a total of 40 points.

1. For each of the following, confirm or refute the assertion. Justify your answer.
   (a) If $\varphi$ is unsatisfiable and $\psi$ is valid, then $(\varphi \rightarrow \psi)$ is valid.
   (b) If $(\varphi \rightarrow \psi)$ is valid, then $\varphi$ is unsatisfiable or $\psi$ is valid.
   (c) If $(\varphi \rightarrow \psi)$ is unsatisfiable, then $\varphi$ is valid and $\psi$ is unsatisfiable.

2. Show that the sentence
$$(\exists z)(\forall x)(\forall y)((f(x) = f(y) \rightarrow x = y) \wedge f(x) \neq z)$$
   does not have any finite model.

3. Consider a countable vocabulary $\Sigma$, and let $T$ be a *theory* over $\Sigma$ (recall that a *theory* is a set of sentences closed under logical implication.) Recall that, by the Lowënheim-Skolem Theorem, if a set of sentences over $\Sigma$ is satisfiable, then it has a countable model. $T$ is *complete* if for every sentence $\varphi$ we have that either $\varphi \in T$ or $\neg\varphi \in T$. $T$ is $\omega$-categorical if all its countable models are isomorphic. Show that if $T$ is $\omega$-categorical and has no finite models, then it is complete.

4. Consider a vocabulary $\Sigma$ for graphs, which consists of a single binary relation symbol $\mathbf{E}$ (the edge relation). A *perfect matching* of a directed graph $G = (V, E)$ is a subset $E' \subseteq E$ of edges such that every node falls on some edge in $E'$ and no two edges in $E'$ share a node. Write a second-order sentence $\varphi$ over $\Sigma$ such that $\varphi$ is true for a graph $G = (V, E)$ precisely when $G$ has a perfect matching. Explain why $\varphi$ is a correct solution.