Logic B-Exam
May 1996

There are 4 questions worth 10 points each for a total of 40 points.

1. Let $\mathsf{add\text{-}carry}$ be a $4n$-ary connective such that $\mathsf{add\text{-}carry}(\mathbf{p}, \mathbf{q}, \mathbf{c}, \mathbf{r})$ is true, for $\mathbf{p} = \langle p_1, \ldots, p_n \rangle$, $\mathbf{q} = \langle q_1, \ldots, q_n \rangle$, $\mathbf{c} = \langle c_1, \ldots, c_n \rangle$, and $\mathbf{r} = \langle r_1, \ldots, r_n \rangle$, when $\mathbf{r}$ is the sum of $\mathbf{p}$ and $\mathbf{q}$ (modulo 2^n) with $\mathbf{c}$ being the carry. Write a $\{\neg, \wedge, \vee\}$-formula of size $O(n)$ that is logically equivalent to $\mathsf{add\text{-}carry}(\mathbf{p}, \mathbf{q}, \mathbf{c}, \mathbf{r})$.

2. A *propositional theory* is a set of propositional formulas closed under logical implication. If S is a set of propositional formulas, then $Th(S) = \{\varphi : S \models \varphi\}$, i.e., $Th(S)$ is the closure of S under logical implication. Thus, T is a propositional theory if $T = Th(T)$.

   Exhibit a strictly increasing sequence $T_1 \subset T_2 \subset T_3 \ldots$ (i.e., T_i is a proper subset of T_{i+1}, for $j \geq 1$) of satisfiable theories.

3. Use compactness to show that if a set Σ of sentences has arbitrarily large finite models then it has an infinite model.

4. Consider a vocabulary $\mathcal{L}$ for graphs that consists of a single binary relation E (the edge relation). Write a second-order sentence φ over $\mathcal{L}$ such that φ is true for a graph $G = (V, E)$ precisely when G is 3-colorable.