Qualifying Examination
in
Foundations
May 17, 1994

The examination consists of 6 problems totaling 95 points. The problems *do not* have equal weight. Budget your time accordingly.

1. (10 pts.) Prove that the following formula holds for any natural number n and any non-zero real number a:

$$\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}.$$

2. (20 pts.) In the following question, let G_1, G_2, ... be an effective enumeration of all context free grammars and let M_1, M_2, ... be an effective enumeration of all Turing machines. Determine whether or not the following sets are *(i)* recursive, *(ii)* r.e. (recursively enumerable), or *(iii)* not r.e. Justify your answers.

   (a) The set $\{i|\ L(G_i)\ is\ finite\}$.

   (b) The set $\{i|\ L(M_i)\ is\ finite\}$.

3. (15 pts.) Let $L = (0+1)^*10(00)^*1(0+1)^*$, the language with at least one pair of 1's separated by an odd number of 0's.

   R_L is equivalence relation over $(0+1)^*$ defined by

$$x\ R_L\ y \quad \Leftrightarrow\ \forall z[xz \in L \Leftrightarrow yz \in L]$$

   Find the equivalence classes of R_L and express your answers as regular expressions.
   Hint: Find the minimum DFA for L.

   Show all the major steps in deriving your answer.

4. (10 pts.) Let $N_\perp$ be the Scott-domain consisting of the natural numbers N and the divergent element $\perp$ under the usual approximation ordering:

$$x \sqsubseteq y \ \Leftrightarrow\ x = \perp\ or\ x = y$$

   Let F be $N_\perp \to N_\perp$, the Scott-domain of continuous functions over $N_\perp$. How does the cardinality of F compare with the cardinality of the set of computable functions from $N_\perp$ to $N_\perp$? Justify your answer.

5. (15 pts.) A term M in the $\lambda - calculus$ has the form defined by the grammar:

$$M \ ::= \ V \mid (M\ M) \qquad (1)$$
$$V \ ::= \ x \mid (\lambda x.M) \qquad (2)$$

where x ranges over a set of variables. The β axiom is

$$((\lambda x.M)N) = M[N/x]$$

where $M[N/x]$ is the capture-avoiding substitution of N for the free occurrences of x in M. The resulting equational theory, is called $\boldsymbol{\lambda}$.

There is a variation on the β axiom, called the β-value axiom, that yields a different equational theory over the terms of the $\lambda - calculus$. The β-value axiom has the form:

$$((\lambda x.M)V) = M[V/x]$$

where V is restricted to the form given the grammar above and $M[V/x]$ is the capture-avoiding substitution of V for the free occurrences of x in M. The resulting equational theory is called $\boldsymbol{\lambda}_v$.

Both theories are Church-Rosser. Find two terms that are equivalent in $\boldsymbol{\lambda}$ but not in $\boldsymbol{\lambda}_v$. Justify your answer.

6. (25 pts.) A *graph* G is a pair (V, E) consisting of a (possible infinite) nonempty set of nodes V and a binary relation $E \subseteq V^2$. G is *finite* if V is finite. A *path of length* k in G between nodes u and v is a sequence $u_0, ..., u_k$ of nodes in V where $k > 0$ such that $u_0 = u, u_k = v$, and $(u_i, u_{i+1}) \in E$ for $0 \leq i < k$. In this case, we say that the *distance* between u and v is at most k. G is *connected* if there is a path between every ordered pair of distinct nodes u and v in V.

A graph *vocabulary* consists of a binary predicate symbol $\overline{E}$ whose intended interpretation is the binary relation E. Thus, we can express graph properties using first-order sentences over the graph vocabulary. For example, the first-order sentence $\forall x \exists y(\overline{E}(x, y) \vee \overline{E}(y, x))$ says that there are no isolated nodes in G. A graph property is *first-order definable* if there is a first-order sentence φ_P (with equality) over the graph vocabulary whose set of models is precisely the class of graphs satisfying the property P.

(a) Show that the first-order sentence

$$\forall x, y, z \exists u(\overline{E}(x, u) \wedge (\overline{E}(x, y) \wedge \overline{E}(y, z) \rightarrow \overline{E}(x, z)) \wedge \neg \overline{E}(x, x))$$

has no finite models.

(b) Describe a formula $\varphi(x, y)$ expressing the fact that the distance between x and y is at most k for $k > 0$.

(c) Prove that connectivity is not first-order definable.