Qualifying Examination
in
Foundations

Jan 12, 1994

The examination consists of seven problems. All the problems have equal weight. Budget your time accordingly.

1. Let N be a set defined recursively as:

   (base) $0 \in N$.

   (rec) $s(x) \in N$ for $x \in N$.

   So, $N = \{0, s(0), s(s(0)), ...\}$. Define the function $A : N \times N \rightarrow N$ by:

   (base) $A(0, y) = y$ for $y \in N$.

   (rec) $A(s(x), y) = s(A(x, y))$ for $x, y \in n$.

   Prove the following statements about A and s.

   (a) $A(x, s(y)) = s(A(x, y))$ for all $x, y \in N$.

   (b) $A(x, y) = A(y, x)$ for all $x, y \in N$.

2. Let M_i denote the partial function computed by the ith Turing machine in a standard enumeration $M_0, M_1, \ldots$ of Turing machines. Is the set

$$\{i | M_i(x) \text{ diverges for all inputs } x\ \}$$

   recursive? Is it recursively enumberable? Justify your answers without citing Rice's theorems.

3. Is the language

$$L = \{a^n b^m | n \leq m \leq 2n\}$$

   regular? Is it context-free? Justify your answers.

4. Let L be a lattice. An element $x \in L$ is *compact* (or *finite*) *iff* for all sets $A \subseteq L$ such that $x \sqsubseteq \bigsqcup A$ there exists *finite* subset $B \subseteq A$ such that $x \sqsubseteq \bigsqcup B$. Either prove or disprove (by giving a counterexample) the following statement: if a and b are compact elements of L, $a \sqcup b$ is compact. (The notation $\sqcup A$ means the "least upper bound of A". Similarly, the notation $a \sqcup b$ means the the "least upper bound of a and b".)

5. Define a function that is continuous but not computable.

6. Construct two *closed* terms M and N in the λ-calculus that are $\beta\eta$ equivalent ($M =_{\beta\eta} N$) but not β equivalent ($M =_{\beta} N$). Recall the η conversion rule:

$$M = \lambda x.Mx$$

where the usual restrictions on capture apply.

7. In a first order language *without equality*, the following theorem (proved by Lowenheim) holds: if a set A of sentences has a finite model with k elements, then it has a finite model with n elements for every natural number $n > k$. Does the same theorem hold for first order languages *with equality*? Prove it or give a counterexample.