Lecture 4: More Semantics

1 Lecture Overview

Recall the two levels of logic - *syntax* and *semantics*. While syntax deals with the form or structure of the language, it is semantics that adds meaning to the form.

There is a famous painting by the French painter Rene Magritte, which shows a pipe, and underneath it says "Ceci n'est pas une pipe", meaning, "this is not a a pipe". What Magritee meant was that there is distinction between a pipe and a picture of a pipe. That is exactly the distinction between syntax and semantics

Ludwig Witgenstein was an important 20th Century phisolopher. His book, *Tractatus Logico-Philosophicus*, starts with:

```
1 The world is all that is the case.
1.1 The world is the totality of facts, not of things.
1.11 The world is determined by the facts, and by their being all
the facts.
1.12 For the totality of facts determines what is the case, and also
whatever is not the case.
1.13 The facts in logical space are the world.
1.2 The world divides into facts.
1.21 Each item can be the case or not the case while everything else
remains the same.
```

The set of facts of the world is precisely our set $Prop$ of atomic propositions. The set of formulas is the assertion we make about the world. Semantics bridges the syntax and the world.

In the last lecture we introduced the *philosopher's perspective* and the *electrical engineer's perspective* of semantics associated with formulas in propositional logic. In this lecture we finish our discussion of semantics by introducing the *software engineer's perspective*.

1.1 Semantics

We first define a *world* as any truth assignment $\tau : Prop \to \{0, 1\}$. Equivalently, $\tau \in 2^{Prop}$. A world may also be called a *semantical domain*, or from the philosophical perspective, a *domain of discourse*.

We have built a model for syntax and a model for the world. Now we may use semantics as a bridge between them. Recall the philosopher's view from last lecture, which uses a binary relation $\models \subseteq 2^{Prop} \times Form$. Then, given a truth assignment $\tau \in 2^{Prop}$ and a formula $\varphi \in Form$ such that $\tau \models \varphi$, it is said that *τ satisfies φ*.

We have also considered the electrical engineer's view, under which a formula $\varphi \in Form$ can be viewed as a boolean function (or circuit) that maps from 2^{Prop} to $\{0, 1\}$.

We concluded last lecture with a proof that the philosopher's approach and the electrical engineer's approach are equivalent.

2 Software Engineer's View

We introduce an additional model of semantics, called the software engineer's approach. It serves as a set-theoretic view, where a formula defines the set of assignments that make the formula true. To this end, we define the function $models : Form \to 2^{2^{Prop}}$ so that for all $p \in Prop$ and $\varphi, \psi \in Form$,

1. $models(p) = \big\{\tau \in 2^{Prop} \mid \tau(p) = 1\big\}$
2. $models((\neg\varphi)) = 2^{Prop} \setminus models(\varphi)$
3. $models((\varphi \wedge \psi)) = models(\varphi) \cap models(\psi)$
4. $models((\varphi \vee \psi)) = models(\varphi) \cup models(\psi)$
5. $models((\varphi \to \psi)) = \big(2^{Prop} \setminus models(\varphi)\big) \cup models(\psi)$
6. $models((\varphi \leftrightarrow \psi)) = ((2^{Prop} \setminus models(\varphi)) \cap (2^{Prop} - models(\psi))) \cup (models(\varphi) \cap models(\psi))$

Then for each $\varphi \in Form$, $models(\varphi)$ is precisely the set of all worlds in which φ is true.

We must now prove that the software engineer's model of semantics is equally as powerful as the previous two approaches.

Lemma 1. *For all $\varphi \in Form$, $models(\varphi) = \big\{\tau \in 2^{Prop} \mid \varphi(\tau) = 1\big\}$.*

Proof. We proceed by structural induction on φ. In the base case, for any $p \in Prop$, we have that

$$models(p) = \{\tau \mid \tau(p) = 1\} = \{\tau \mid p(\tau) = 1\}$$

by definition. Let $\varphi, \psi \in Form$. Then

$$\begin{aligned} models\,((\neg\varphi)) &= 2^{Prop} \setminus models(\varphi) \\ &= 2^{Prop} \setminus \{\tau \mid \varphi(\tau) = 1\} \\ &= \{\tau \mid \varphi(\tau) = 0\} \\ &= \{\tau \mid (\neg\varphi)\,(\tau) = 1\}. \end{aligned}$$

Also,

$$\begin{aligned} models(\varphi \wedge \psi) &= models(\varphi) \cap models(\psi) \\ &= \{\tau \mid \varphi(\tau) = 1\} \cap \{\tau \mid \psi(\tau) = 1\} \\ &= \{\tau \mid \varphi(\tau) = \psi(\tau) = 1\} \\ &= \{\tau \mid \wedge(\varphi(\tau), \psi(\tau)) = 1\}. \end{aligned}$$

Finally,

$$\begin{aligned} models(\varphi \vee \psi) &= models(\varphi) \cup models(\psi) \\ &= \{\tau \mid \varphi(\tau) = 1\} \cup \{\tau \mid \psi(\tau) = 1\} \\ &= \{\tau \mid \varphi(\tau) = 1 \text{ or } \psi(\tau) = 1\} \\ &= \{\tau \mid \vee(\varphi(\tau), \psi(\tau)) = 1\}. \end{aligned}$$

The cases involving the other binary connectives can be proven similarly. □

3 Relevance Lemma

In propositional logic, information that is extraneous to a formula does not affect its truth value. For example, the transitivity formula

$$\varphi = ((p \to q) \to ((q \to r) \to (p \to r))).$$

is true for any world in which the propositions p, q, and r are true. In particular, $\{p, q, r\} \models \varphi$ and $\{p, q, r, s\} \models \varphi$. Since the proposition s does not occur in the formula φ, the truth value of s does not affect the truth value of φ.

To formally prove this fact, we first require a well-defined notion of *occurrence.*

Definition 1 (AP(φ))**.** *For $\varphi \in Form$, the set $AP(\varphi)$ of atomic propositions that occur in φ is defined as follows:*

1. *$AP(p) = p$, where $p \in Prop$.*
2. *$AP((\neg\varphi)) = AP(\varphi)$, where $\varphi \in Form$.*
3. *$AP((\varphi \circ \psi)) = AP(\varphi) \cup AP(\psi)$, where $\varphi, \psi \in Form$.*

We are now ready to state and prove the informal result from above.

Lemma 2 (Relevance lemma)**.** *Let $\varphi \in Form$ with $\tau, \tau' \in 2^{Prop}$. If $\tau \cap AP(\varphi) = \tau' \cap AP(\varphi)$, then $\varphi(\tau) = \varphi(\tau')$.*

Proof. Let $p \in Prop$. Then $\tau \cap AP(p) \subseteq \{p\}$. If $\tau \cap AP(p) = \tau' \cap AP(p) = \varnothing$, then $p(\tau) = p(\tau') = 0$. Otherwise, $\tau \cap AP(p) = \tau' \cap AP(p) = \{p\}$, and so $p(\tau) = p(\tau') = 1$.

Now let $\varphi, \psi \in Prop$. If

$$\tau \cap AP((\neg\varphi)) = \tau' \cap AP((\neg\varphi)),$$

then

$$\tau \cap AP((\varphi)) = \tau' \cap AP((\varphi)),$$

which, by the inductive hypothesis, implies that $\varphi(\tau) = \varphi(\tau')$. Then, by definition, $(\neg\varphi)(\tau) = (\neg\varphi)(\tau')$. If

$$\tau \cap AP((\varphi \circ \psi)) = \tau' \cap AP((\varphi \circ \psi)),$$

then, by the inductive hypothesis, we have that

$$\circ(\varphi(\tau), \psi(\tau)) = \circ(\varphi(\tau'), \psi(\tau')).$$

□

The importance of the Relevance Lemma will become apparent in the next few lectures when we discuss satisfiability. Instead of talking about the infinite set of truth assignments, it allows us to focus on the finite set of propositions that occur in a logical formula.