# Lecture 3: Semantics of Propositional Logic

## 1 Semantics of Propositional Logic

Every language has two aspects: *syntax* and *semantics*. While syntax deals with the form or structure of the language, it is semantics that adds meaning to the form. The words or expressions of the language are interpreted with respect to some world. Semantics is a mapping from the language to that world. In the case of propositional logic, the words of the language are the formulas in FORM and the world in which the formulas should be interpreted is given by a truth assignment $\tau : \text{PROP} \to \{0, 1\}$, which is a function that assigns true or false to each atomic proposition in PROP.

Now a function that assigns 0 or 1 to each element of a set can also be viewed as selecting a subset of the set: if an element is assigned value 1 then add it to the subset, otherwise leave it out. Thus, such a function can be uniquely identified with a subset of its domain. With this intuition, which we formalize below, each subset of PROP can be viewed as a truth assignment. The set of all truth assignments, which is also called the set of *possible worlds*, the *semantical domain*, or the *domain of discourse*, is $2^{\text{PROP}}$. *Semantics* is the relation between FORM, the set of all formulas, and $2^{\text{PROP}}$, the set of all worlds. We call this relation *satisfaction*, denoted by $\models$, where $\models \subseteq 2^{\text{PROP}} \times \text{FORM}$.

**Definition 1.** *A truth assignment, $\tau$, is an element of* $2^{\text{PROP}}$.

There are two ways to think of truth assignments:

1. $2^{\text{PROP}}$ can be thought of as the power set of all propositions, and a truth assignment $X$ is an element of it, that is, $X \subseteq \text{PROP}$. Accordingly, a proposition is true if it is in the subset $X$ of $2^{\text{PROP}}$ and is otherwise false.

2. Or, we can think of $2^{\text{PROP}}$ as the set of all functions from PROP to the set $\{0, 1\}$. Then, a truth assignment is a map $\tau : \text{PROP} \mapsto \{0, 1\}$. A proposition is true if it maps to 1.

These notions are equivalent since given $X \subseteq \text{PROP}$, we can define $\tau_X : \text{PROP} \mapsto \{0, 1\}$ by

$$\tau_X(p) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{if } p \notin X \end{cases}$$

which is the characteristic function of $X$. Conversely, given $\tau : \text{PROP} \mapsto \{0,1\}$, we can take $X_\tau = \{p | \tau(p) = 1\}$.

Recall from the first lecture, that logic is used in many different fields including philosophy, foundational mathematics, computer science and electrical engineering. Each of these fields bring their own approach to logic based upon their different needs and motivations. For example, while philosophers use logic to reason about fundamental concepts such as truth and knowledge, electrical engineers use logic to build a mathematical foundation for their theory of electronic circuits. Here we look at how the semantics of propositional logic are defined by different fields that use logic. We will look at three different perspectives:

1. philosopher's view
2. electrical engineer's view
3. software engineer's view

We will see that despite their very different approaches to the subject, the different definitions are equivalent.

# 2 Philosopher's view

For a philosopher, semantics is a binary relation $\models$ between structures and formulas. $\tau \models \varphi$ means:

1. $\tau$ satisfies $\varphi$ or
2. $\tau$ is true of $\varphi$ or
3. $\varphi$ holds at $\tau$ or
4. $\tau$ is a model of $\varphi$.

**Example:**
object $\tau$: table
formular $\varphi$: this table is round
$\tau \models \varphi$ can be either true of false

**Definition 2.** $\models \subseteq (2^{\text{PROP}} \times \text{FORM})$ *is a binary relation, between truth assignments and formulas.* $\models$ *is called the* satisfaction relation, semantical relation*, or the* truth relation. *We define it inductively as follows:*

- $\tau \models p$, for $p \in \text{PROP}$, if $\tau(p) = 1$, meaning that $p$ holds if $p$ is true.
- $\tau \models (\neg\varphi)$ if it is not the case that $\tau \models \varphi$, that is, if $\tau \not\models \varphi$ (Note: this is so only in 2-valued world).

- $\tau \models (\varphi \vee \psi)$ if $\tau \models \varphi$ or $\tau \models \psi$.
- $\tau \models (\varphi \wedge \psi)$ if $\tau \models \varphi$ and $\tau \models \psi$.
- $\tau \models (\varphi \rightarrow \psi)$ if $\tau \models \varphi$ implies $\tau \models \psi$, i.e. if $\tau \not\models \varphi$ or $\tau \models \psi$.
- $\tau \models (\varphi \leftrightarrow \psi)$ if $\tau \models \varphi$ iff $\tau \models \psi$.

It is important to realize that the semantics is well defined because of the Unique Readability Theorem. The definition above relies on the fact that, given a formula, we can talk about its primary connective and its immediate subformula(s) without ambiguity. Without unique readability, we would not have well-defined semantics. Suppose for example that we do not require parentheses, and consider the "formula" $p \wedge q \vee r$. Let $\tau$ be a truth assignment such that $\tau(p) = 0$, $\tau(q) = 1$, and $\tau(r) = 1$. Since the example formula does not have a unique primary connective, its meaning may be defined in the following two ways:

- $\tau \models p \wedge q \vee r$ iff $\tau \models p$ and $\tau \models q \vee r$. Since $\tau \not\models p$, we have that $\tau \not\models p \wedge q \vee r$.
- $\tau \models p \wedge q \vee r$ iff $\tau \models p \wedge q$ and $\tau \models r$. Since $\tau \models r$, $\tau \models p \wedge q \vee r$.

According to the above example, a formula could have two different truth values if it did not have a unique primary connective. A similar kind of ambiguity exists also in natural languages. Consider the following English phrase: "John hates Jim and he likes Jill". Who is he? It is ambiguous.

This points out an important feature of this definition, the *compositionality* of $\models$. An example is $\tau \models (\varphi \wedge \psi)$. All that is needed is information about $\tau \models \varphi$ and $\tau \models \psi$ and those answers are independent of each other. This is alot like modularity in programming.

# 3 Electrical Engineer view

To an electrical engineer the truth assignment is simply a mapping of voltages on a wire: $\tau : Prop \rightarrow \{0,1\}$

Operations are carried out by gates, which represent logical connectives. The value of the voltage along the wire is continuous, hence a zero value is assigned to an interval of voltages and a one is assigned to a non-overlapping interval of voltage. An AND gate takes in two voltages and returns a voltage representing 1 if and only if both input voltages represented a 1. In general, a k-ary gate $g$ is a mapping $g : \{0,1\}^k \rightarrow \{0,1\}$. It takes in voltages and returns a voltage as dictated by its logical specification. Here are some examples:

- $\neg : \{0,1\} \rightarrow \{0,1\}$
  - $\neg(0) = 1$ and $\neg(1) = 0$
- $\wedge : \{0,1\}^2 \rightarrow \{0,1\}$

– $\wedge(0,0) = \wedge(1,0) = \wedge(0,1) = 0$ and $\wedge(1,1) = 1$

- $\vee : \{0,1\}^2 \to \{0,1\}$
  – $\vee(1,1) = \vee(1,0) = \vee(0,1) = 1$ and $\vee(0,0) = 0$
- $\to: \{0,1\}^2 \to \{0,1\}$
  – $\to (1,1) =\to (0,0) =\to (0,1) = 1$ and $\to (1,0) = 0$
- $\leftrightarrow: \{0,1\}^2 \to \{0,1\}$
  – $\leftrightarrow (1,1) =\leftrightarrow (0,0) = 1$ and $\leftrightarrow (1,0) =\leftrightarrow (0,1) = 0$

We can now think of a formula as a circuit, which maps truth assignments to Boolean values: $\varphi : 2^{Prop} \to \{0,1\}$.

The logical operands, defined previously in the section on logic in philosophy, can be alternatively defined as gates that take in the inputs $\varphi$ and $\psi$ and returns the specified output.

| $\varphi$, | $\psi$ | $(\neg\varphi)$ | $(\varphi \wedge \psi)$ | $(\varphi \vee \psi)$ | $(\varphi \to \psi)$ | $(\varphi \leftrightarrow \psi)$ | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | $\star$ the $\neg$ takes a single |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | immediate subformula |
| 0 | 1 | $\star$ | 1 | 0 | 1 | 0 | and has only two |
| 1 | 1 | $\star$ | 0 | 1 | 1 | 1 | possible inputs. |

**Definition 3.** *Let $p \in Prop, \tau \in 2^{Prop}$. Then the semantics is defined according to the following rules:*

- $p(\tau) = \tau(p)$ *(a proposition viewed as a circuit is just a wire)*
- $(\neg\varphi)(\tau) = \neg(\varphi(\tau))$
- $(\varphi \circ \psi)(\tau) = \circ(\varphi(\tau), \psi(\tau))$

The first rule says that $\tau$ maps a proposition to its truth value. The second and third rule say that with a given $\tau$ the value of a formula can be determined from the truth values of its immediate subformulae by applying the primary connective, its functionality defined in the above table, to the values of the immediate subformulas under the same $\tau$.

Next we prove that these two semantic formulations are equivalent. The transition between each step in the proof uses equivalence, meaning that the steps can be reversed to prove the only if part of the lemma. Notice the almost mechanical pattern of the proof. In each case the philosopher's definition is applied first, followed by the inductive hypothesis, followed by application of the above table, followed by application of the electrical engineer's definition of the connective.

**Lemma 1.** *Let $\varphi \in Form$ and $\tau \in 2^{Prop}$. Then $\tau \models \varphi$ iff $\varphi(\tau) = 1$.*

*Proof.* Proof by structural induction:

**Basis** : $\varphi = p$, for $p \in \mathrm{PROP}$. Then $\tau \models \varphi$ iff $\tau(p) = 1$ iff $p(\tau) = 1$.

**Inductive Step** :

1. $\varphi = (\neg\psi)$. Then $\tau \models (\neg\psi)$ iff $\tau \not\models \psi$ iff $\psi(\tau) = 0$ iff $\neg(\psi(\tau)) = 1$ iff $(\neg\psi)(\tau) = 1$.
2. $\varphi = (\psi \wedge \theta)$. Then $\tau \models (\psi \wedge \theta)$ iff ($\tau \models \psi$ and $\tau \models \theta$) iff $\psi(\tau) = 1$ and $\theta(\tau) = 1$ iff $\wedge(\psi(\tau), \theta(\tau)) = 1$ iff $(\psi \wedge \theta)(\tau) = 1$.
3. $\varphi = (\psi \vee \theta)$. Then $\tau \models (\psi \vee \theta)$ iff ($\tau \models \psi$ or $\tau \models \theta$) iff $\psi(\tau) = 1$ or $\theta(\tau) = 1$ iff $\vee(\psi(\tau), \theta(\tau)) = 1$ iff $(\psi \vee \theta)(\tau) = 1$.
4. $\varphi = (\psi \to \theta)$. Then $\tau \models (\psi \to \theta)$ iff ($\tau \not\models \psi$ or $\tau \models \theta$) iff $\psi(\tau) = 0$ or $\theta(\tau) = 1$ iff $\to(\psi(\tau), \theta(\tau)) = 1$ iff $(\psi \to \theta)(\tau) = 1$.

$\square$