# Lecture 26

## 1 Herbrand Universes and Structures

If $\varphi$ is satisfiable, how can we find a structure $A$ that satisfies $\varphi$? We will use the syntax as our domain.

Consider a vocabulary which contains at least one constant symbol (and without equality).

**Definition 1 (Ground Term)** *A* ground term *is a term containing no variables.*

We can think of a ground term as an "partially evaluated" term in the sense that the variable will be replaced with an object in the domain.

**Definition 2 (Herbrand Universe)** *The* Herbrand Universe *(denoted $H$) of a language is the set of ground terms in a language.*

**Definition 3 (Herbrand Structure)** *A* Herbrand Structure *of a formula $\varphi$ is a structure $\mathcal{H}$ such that $D^{\mathcal{H}}$ is the Herbrand Universe of $\varphi$ and $f^{\mathcal{H}}([t_1]\dots[t_n]) = [f(t_1\dots t_n)]$.*

I.e.,

$$\mathcal{H} = (D^{\mathcal{H}}, P_1^{\mathcal{H}} \dots P_k^{\mathcal{H}}, f_1^{\mathcal{H}} \dots f_\ell^{\mathcal{H}})$$

where $D^{\mathcal{H}}$ is an *Herbrand Universe*. Terms are now both syntax and semantics: $f_i^{\mathcal{H}}(``t_1", \dots, ``t_n") = ``f_i(t_1, \dots, t_n)"$.

**Definition 4 (Herbrand Model)** *An* Herbrand Model *$\mathcal{H}$ of a sentence $\varphi$ is an Herbrand structure that satisfies $\varphi$.*

**Theorem 1 (Herbrand's Theorem)** *A universal sentence $\varphi$ without identity is satisfiable iff it is satisfiable in a Herbrand model. Alternatively: If $\varphi$ is a satisfiable sentence, then it has an Herbrand model.*

**Conclusion** Every Herbrand model is countable. In particular, if it is infinite it is countably infinite.

**Corollary 1 (Löwenheim-Skolem Theorem)** *A first order sentence $\varphi$ is satisfiable iff it is satisfiable in a countable structure. Alternatively: If $\varphi$ is satisfiable, then it has a countable model.*

**Corollary 2** *To check if $\varphi$ is valid, we only have to consider countable structures.*

**Corollary 3** *There is no first order sentence to express the notion "There are uncountably many elements."*

## 2 Validity is Recursively Enumerable

**Definition 5** *Let $\varphi$ be a universal sentence $\forall x_1 \ldots \forall x_n \theta$. Then,*

$$ground(\varphi) = \{\theta[x_1 \mapsto t_1, \ldots, x_n \mapsto t_n] \,:\, t_i \in H\}.$$

Note that elements of $ground(\varphi)$ have no variables.

**Lemma 1** *Let $A$ be an Herbrand structure. Then $A \models \varphi$ iff $A \models ground(\varphi)$.*

**Proof.** $A \models \varphi$ iff for all $t_1, \ldots, t_k \in H$ we have $A, [x_1 \mapsto t_1, \ldots, x_k \mapsto t_k] \models \varphi$. By a simple inductive argument on the structure of $\varphi$ we can show that $A, [x_1 \mapsto t_1, \ldots, x_k \mapsto t_k] \models \varphi$ iff $A \models \varphi[x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$. □

**Definition 6** *For every $k$-ary relation symbol $P$ and $t_1, \ldots, t_k \in H$, we introduce a proposition $P_{t_1,\ldots,t_k}$.*

This gives us the following result:

**Lemma 2** *$ground(\varphi)$ is satisfiable in an Herbrand structure iff $ground(\varphi)[p(t_1, \ldots, t_k) \mapsto p_{t_1,\ldots,t_k}$ is satisfiable.*

For example, $p(f(c)) \wedge \neg p(g(d))$ becomes $P_{f(c)} \wedge \neg P_{g(d)}$.

Thus, $\varphi$ is unsatisfiable iff $ground(\varphi)$ is unsatisfiable. It follows from the Compactness Theorem that some finite subset of $ground(\varphi)$ is unsatisfiable. In this way, we have reduced a formula in first order logic to one in propositional logic. In conclusion, to determine the unsatisfiability of $\varphi$, we enumerate all finite subsets of $ground(\varphi)$ and check for unsatisfiability. Finally, recall that $\varphi$ is valid iff $\neg\varphi$ is unsatisfiable.

**Theorem 2 (Gödel's Completeness Theorem)** *Validity for first order logic is recursively enumerable.*

## 3 Validity and Satisfiability in the Finite

Now, what about validity over all finite structures. Unfortunately, there we have another negative result.

**Definition 7** *$\varphi$ is valid in the finite if $A \models \varphi$ holds whenever $A$ is finite.*

**Theorem 3 (Trakhtenbrot's Theorem)** *Validity over finite structures for first-order logic is not recursive.*

**Definition 8** *$\varphi$ is satisfiable in the finite if there is a finite structure $A$, such that $A \models \varphi$*

What about satisfiability in a finite structure? If the vocabulary is finite, then there are finitely many structures (up to an isomorphism) of a given finite cardinality.

**Corollary 4** *Satisfiability in some finite structure is recursively enumerable.*

Hence, validity over finite structures cannot be recursively enumerable.