

## Lecture 22 - FOL Expressiveness - III

### 1 Definability of a class of structures: $EC$ and $EC_{\Delta}$

**Definition 1** ( $EC$ ). A class  $C$  of structures is elementary, denoted  $EC$ , if  $C = \text{models}(\varphi)$  for some sentence  $\varphi$ .

**Example 1.** We give some examples of  $EC$  classes, along with the sentences that define them.

- The class of undirected graphs:

$$\varphi_1 = (\forall x)(\neg E(x, x)) \wedge (\forall x)(\forall y)(E(x, y) \leftrightarrow E(y, x))$$

- The class of triangle free undirected graphs:

$$\varphi_2 = \varphi_1 \wedge ((\forall x)(\forall y)(\forall z)(E(x, y) \wedge E(y, z) \rightarrow \neg E(x, z)))$$

- The class of partial orders:

$$\begin{aligned} \varphi_3 = & ((\forall x)(xRx)) \wedge ((\forall x)(\forall y)(xRy \wedge yRx \rightarrow x \approx y)) \\ & \wedge ((\forall x)(\forall y)(\forall z)(xRy \wedge yRz \rightarrow xRz)) \end{aligned}$$

- The class of total orders:  $\varphi_4 = \varphi_3 \wedge ((\forall x)(\forall y)(xRy \vee yRx))$
- The class of groups:

$$\begin{aligned} \varphi_5 = & (\forall x)(x + 0 \approx x \wedge 0 + x \approx x) \wedge (\forall x)(\exists y)(x + y \approx 0 \wedge y + x \approx 0) \\ & \wedge (\forall x)(\forall y)(\forall z)((x + y) + z \approx x + (y + z)) \end{aligned}$$

- The class of abelian groups:  $\varphi_6 = \varphi_5 \wedge (\forall x)(\forall y)(x + y \approx y + x)$

We note that any mathematical structure that can be defined using a finite number of axioms (e.g., groups, rings, vector spaces, boolean algebras, topological spaces) can also be defined using a single sentence, by taking the finite conjunction of the axioms. This naturally leads us to ask whether we would gain any expressiveness by allowing infinite sets of sentences to define classes of structures.

**Definition 2** ( $EC_\Delta$ ). A class  $C$  of structures is extended elementary, denoted  $EC_\Delta$ , if  $C = \text{models}(\Phi)$  for some set of sentences  $\Phi$ .

Note that, since a finite set of sentences can be written as a single finite conjunction, the real expressive power of the above definition comes from allowing infinite number of sentences.

Our first example of an  $EC_\Delta$  class is the class of infinite structures (i.e., structures with an infinite domain), denoted  $C_{\text{INF}}$ .

**Theorem 1.** The class of infinite structures,  $C_{\text{INF}}$ , is  $EC_\Delta$ .

*Proof.* Given a natural number  $n$ , we can define the property that there are at least  $n$  elements in the domain as a first order sentence as follows:

- at least 2:  $\psi_2 = \exists x_1 \exists x_2 (x_1 \neq x_2)$
- at least 3:  $\psi_3 = \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_3 \neq x_1)$
- at least  $k$ :  $\psi_k = \exists x_1 \cdots \exists x_k \bigwedge_{i \neq j} (x_i \neq x_j)$

Consider the set  $\Sigma = \{\psi_n : n \in \mathbb{N}\}$ . If  $A$  is a finite structure, then  $A \not\models \psi_k$  for  $k > |D^A|$ . So  $A \notin \text{models}(\Sigma)$ . If  $A$  is an infinite structure, then  $A$  has at least  $n$  elements for all  $n \in \mathbb{N}$ . So  $A \models \psi_n$  for all  $n \in \mathbb{N}$ , and thus  $A \in \text{models}(\Sigma)$ . Therefore  $\text{models}(\Sigma)$  is exactly the class of infinite structures, and  $C_{\text{INF}}$  is  $EC_\Delta$ .  $\square$

In order to discuss our second example of a class of structures that is  $EC_\Delta$ , recall the following definition of  $k$ -colorable graphs:

**Definition 3** ( $k$ -colorability).  $G = \{V, E\}$  is  $k$ -colorable iff there is a mapping  $c : V \rightarrow \{1, \dots, k\}$  such that  $E(u, v) \rightarrow c(u) \neq c(v)$ .

In the remainder of this lecture we will use  $2C$  to denote the class of 2-colorable graphs. As the following well-known graph-theoretic result shows, 2-colorability has a particularly simple characterization.

**Lemma 1.** A graph is 2-colorable iff it has no cycle of odd length.

The lemma can be proved easily by induction and we leave the proof as an exercise. We will use the lemma to construct an infinite set of sentences that define  $2C$ .

**Theorem 2.**  $2C$  is  $EC_\Delta$ .

*Proof.* By the lemma above,  $2C$  is the class of graphs which have no cycles of odd length. So a graph  $G$  is in  $2C$  iff for all  $n \in \mathbb{N}$ ,  $G$  has no cycle of length  $2n + 1$ . Suppose we can represent the property that a graph has no cycle of length  $k$  as a sentence  $\varphi_k$ , for  $k \in \mathbb{N}$ . Then  $2C$  can be defined by the set of sentences  $\{\varphi_{2n+1} : n \in \mathbb{N}\}$ . We now construct such a sentence.

We first define formulas  $p_n(x, y)$  recursively as follows:

$$p_1(x, y) = E(x, y) \tag{1}$$

$$p_n(x, y) = (\exists z)(p_{n-1}(x, z) \wedge E(z, y)) \tag{2}$$

That is,  $p_n(x, y)$  iff there is a path of length  $n$  from vertex  $x$  to vertex  $y$ . Likewise,  $p_n(x, x)$  iff there is a path of length  $n$  from vertex  $x$  to itself, that is,  $x$  lies on a cycle of length  $n$ .

Then the condition that the graph has a cycle of length  $n$  can be defined by the sentence  $cycle_n$  as follows:

$$cycle_n = (\exists x)(p_n(x, x))$$

Finally, the condition that the graph has no odd cycles is defined by the set of sentences  $\Phi$ , defined as:

$$\Phi = \{\neg cycle_{2n+1} : n \in \mathbb{N}\}$$

By the lemma, we have  $2C = models(\Phi)$  and  $2C$  is  $EC_\Delta$  □

## 2 Relating $EC$ and $EC_\Delta$

**Definition 4** ( $\overline{C}$ ). Let  $\Sigma$  be a vocabulary and  $C$  a class of structures on  $\Sigma$ , then  $\overline{C}$  is the class of structures that are not in  $C$ .

Notice that we don't want  $\overline{C}$  to just be the unrestricted complement of  $C$ , because it will include things that are not well-formed structures too. So we take complement with respect to the class of all structures. In the rest of the lecture when we refer to the complement of  $C$ , we will mean  $\overline{C}$ .

We now consider the relation between  $C$  and  $\overline{C}$  and between  $EC$  and  $EC_\Delta$ .

**Lemma 2.** *Properties of  $EC$ :*

1. If  $C$  is  $EC$ , then  $C$  is  $EC_\Delta$
2. If  $C$  is  $EC$ , then  $\overline{C}$  is also  $EC$ .

*Proof.* 1. A single sentence is also a singleton set of sentences. If  $C = models(\varphi)$  then  $C = models(\{\varphi\})$ , so we have  $EC \Rightarrow EC_\Delta$ .

2. If  $C$  is  $EC$ , then  $C = models(\varphi)$  for some sentence  $\varphi$ . Since  $C$  and  $\overline{C}$  are disjoint, for any structure  $A$ , we have  $A \in \overline{C}$  iff  $A \notin C$  iff  $A \not\models \varphi$  iff  $A \models \neg\varphi$ . Therefore,  $\overline{C} = models(\neg\varphi)$  and  $\overline{C}$  is also  $EC$ . □

Now that we have two notions of classes of structures, elementary and extended elementary, we wish to determine what distinguishes them. We know that any structure that is  $EC$  is also  $EC_\Delta$ , but is the reverse true? Are there classes that are  $EC_\Delta$  but not  $EC$ ? And are  $C_{INF}$  and  $2C$  in  $EC$ ? In order to answer such questions, we turn to a fundamental notion in first order logic: *compactness*.

**Definition 5** (Finite Satisfiability). *A set  $\Phi$  of formulas is finitely satisfiable iff every finite subset  $\Phi'$  of  $\Phi$  is satisfiable.*

**Theorem 3. Compactness Theorem:** *A set of formulas  $\Phi$  is satisfiable iff it is finitely satisfiable.*

We will look at this theorem in more detail in the next lecture. For now, we take it as given and use it to distinguish  $EC$  and  $EC_\Delta$  by showing that  $C_{\text{INF}}$  and  $2C$  are not  $EC$ .

**Theorem 4.** *The class of infinite structures,  $C_{\text{INF}}$ , is not  $EC$ .*

*Proof.* We prove the statement using contradiction. Suppose that  $C_{\text{INF}}$  is  $EC$ . Then there is a sentence  $\varphi$  such that  $C_{\text{INF}} = \text{models}(\varphi)$ . Now, by Theorem 1, we have  $C_{\text{INF}} = \text{models}(\Sigma)$ , where  $\Sigma = \{\psi_n : n \in N\}$ . Therefore,  $\text{models}(\neg\varphi) \cap \text{models}(\Sigma) = \text{models}(\neg\varphi) \cap \text{models}(\varphi) = \emptyset$ , which means the set of sentences  $\Phi = \{\neg\varphi\} \cup \Sigma$  is unsatisfiable. Then, by the compactness theorem,  $\Phi$  is not finitely satisfiable, and so there exists a finite subset  $\Phi'$  of  $\Phi$  such that  $\Phi'$  is not satisfiable. Since  $\Phi'$  is finite, there must exist a  $k \in N$  such that  $\Phi' \subseteq \{\neg\varphi\} \cup \{\psi_n : n \leq k\}$ . Let  $A$  be a structure with  $k + 1$  elements in its domain (for a concrete example, consider the complete graph on  $k + 1$  vertices). Then  $A$  is not infinite, so  $A \not\models \varphi$  and therefore  $A \models \neg\varphi$ . Also  $A$  has at least  $n$  elements for all  $n \leq k$ , so  $A \models \psi_n$  for  $n \leq k$ . But then  $A$  satisfies each element in  $\Phi'$  so  $A$  satisfies  $\Phi'$ , which contradicts the fact that  $\Phi'$  is not satisfiable. Thus,  $C_{\text{INF}}$  is not  $EC$ .  $\square$

**Corollary 1.** *The class of finite structures is not  $EC$ .*

**Theorem 5.**  *$2C$  is not  $EC$ .*

*Proof.* The proof is very similar to the proof for  $C_{\text{INF}}$ . We again use proof by contradiction (this is a common feature of arguments that use compactness). Suppose that  $2C$  is  $EC$ . Then there is some sentence  $\varphi$  such that  $2C = \text{models}(\varphi)$ . Now, by Theorem 2, we also have  $2C = \text{models}(\Phi)$ , where  $\Phi = \{\neg\text{cycle}_{2n+1} : n \in N\}$ . Then the set  $\Psi = \{\neg\varphi\} \cup \Phi$  is unsatisfiable, and by compactness, there is a finite subset  $\Psi'$  of  $\Psi$  such that  $\Psi'$  is unsatisfiable. Since  $\Psi'$  is finite, there exists  $k \in N$  such that  $\Psi' \subseteq \{\neg\varphi\} \cup \{\neg\text{cycle}_{2n+1} : n \leq k\}$ . Then a graph which is a simple cycle of length  $2k + 3$  satisfies  $\Psi'$ , which contradicts the unsatisfiability of  $\Psi'$ . Therefore,  $2C$  is not  $EC$ .  $\square$

So  $EC_\Delta$  is strictly more powerful than  $EC$ . Earlier we saw that  $EC$  is closed under complementation, that is,  $C$  is  $EC$  iff  $\overline{C}$  is  $EC$ . What can we say about  $\overline{C}$  when  $C$  is  $EC_\Delta$ ?

**Definition 6** (co- $EC_\Delta$ ). *A class  $C$  of structures is co- $EC_\Delta$  if  $\overline{C}$  is  $EC_\Delta$ .*

**Example 2.** The class of finite structures is co- $EC_\Delta$ , because the class of infinite structures,  $C_{\text{INF}}$ , is  $EC_\Delta$ . Similarly, the class of graphs that are not 2-colorable is co- $EC_\Delta$ , because  $2C$  is  $EC_\Delta$ .

$P/NP$
$P \stackrel{?}{\subseteq} NP$
$NP \stackrel{?}{\neq} co - NP$
$NP \cap co - NP \stackrel{?}{=} P$

Table 1:  $P$  vs  $NP$

recursive/ $RE$
$R \subseteq RE$
$RE \neq co - RE$
$RE \cap co - RE = R$

Table 2:  $R$  vs  $RE$

We can use analogies from previously studied topics to obtain a better grasp on the relationship between  $EC$ ,  $EC_\Delta$  and  $co-EC_\Delta$ . Let us compare  $EC/EC_\Delta$  with  $P/NP$  and  $R/RE$ . We know little about the relationship between  $P$  and  $NP$ , except that  $P \subseteq NP$ . For example, we don't know if  $P$  equals  $NP$ , or if  $NP$  equals  $co - NP$ . This is depicted in Table 2. In contrast to this state of affairs, we have a much better understanding of the relationship between  $R$  and  $RE$ . Recall that  $RE$  is the set of recursively enumerable languages (also called semi-decidable or semi-computable), and  $R$  is the set of recursive languages (also called decidable or computable). A language is in  $RE$  if there is some turing machine that accepts it in finite time. A language is in  $co-RE$  if there is some turing machine that rejects it in finite time. A language is in  $R$  if there is some turing machine that accepts or rejects it in finite time. From the definitions, it is clear that  $RE \cap co - RE = Recursive$ . The situation is shown in Table 2.

Turning our focus back to  $EC$  and  $EC_\Delta$ , we ask whether our situation is analogous to  $P/NP$  or  $R/RE$ . We recall that we already have the answer to some of the analogous questions. We know that  $EC \subseteq EC_\Delta$ , but the equality does not hold because there are classes in  $EC_\Delta$  which are not  $EC$ . The questions that we need to answer are: Is  $EC_\Delta \cap co - EC_\Delta = EC$ ? If  $EC_\Delta \neq co - EC_\Delta$ ? It turns out that  $EC/EC_\Delta$  relationship is analogous to  $R/RE$  and the answer to both these questions is yes.

**Theorem 6.**  $EC_\Delta \cap co-EC_\Delta = EC$

*Proof.* We first prove the easy direction: Let  $C$  be  $EC$ . Then  $\overline{C}$  is also  $EC$ . So  $\overline{C}$  is  $EC_\Delta$ , and therefore,  $C$  is  $co-EC_\Delta$ . Thus  $C$  is in  $EC_\Delta \cap co-EC_\Delta$ .

Now we prove the harder direction: Let  $C$  be  $EC_\Delta$  and  $C$  also be  $co-EC_\Delta$ . This means that  $\overline{C}$  is  $EC_\Delta$ . Thus, there exist sets of sentences,  $\Phi$  and  $\Psi$ , such that  $C = models(\Phi)$  and  $\overline{C} = models(\Psi)$ . Since no structure belongs to both  $C$  and  $\overline{C}$ , therefore  $\Phi \cup \Psi$  is unsatisfiable. Then, by the compactness theorem, there exist finite  $\Phi' \subseteq \Phi$  and finite  $\Psi' \subseteq \Psi$  such that  $\Phi' \cup \Psi'$  is unsatisfiable.

$EC/EC_{\Delta}$
$EC \subset EC_{\Delta}$
$EC_{\Delta} \neq co - EC_{\Delta}$
$EC_{\Delta} \cap co - EC_{\Delta} = EC$

Table 3:  $EC$  vs  $EC_{\Delta}$

Let  $C' = models(\Phi')$ . Since  $\Phi' \subseteq \Phi$ ,  $C \subseteq C'$  (remember that the smaller set of formulas is less restrictive, therefore it has more models). Suppose  $C \neq C'$ , and let  $A \in C' - C$ . Then  $A \in C'$  and  $A \in \overline{C}$ . This implies that  $A \in models(\Phi') \cap models(\Psi) = models(\Phi' \cup \Psi)$ , which means that  $A$  satisfies  $\Phi' \cup \Psi$ . Thus  $A$  satisfies  $\Phi' \cup \Psi'$ , which contradicts the unsatisfiability of  $\Phi' \cup \Psi'$ .

Therefore, we must have  $C = C' = models(\Phi')$ . Since  $\Phi'$  is finite, we can construct a single sentence  $\varphi$  which is the conjunction of the elements of  $\Phi'$ . Then  $models(\varphi) = models(\Phi') = C$ , and therefore  $C$  is  $EC$ .  $\square$

**Corollary 2.**  $EC_{\Delta} \neq co-EC_{\Delta}$ .

**Corollary 3.** If  $C$  is not  $EC$  and  $C$  is  $EC_{\Delta}$ , then  $\overline{C}$  is not  $EC_{\Delta}$ .

The above corollary allows us to prove that something is not  $EC_{\Delta}$ .

**Example 3.** The class of finite structures is not  $EC_{\Delta}$ . This is because  $C_{INF}$  is  $EC_{\Delta}$  but is not  $EC$ . Similarly, the class of graphs that are not 2-colorable is not  $EC_{\Delta}$ .

Review: Given a class of structures,  $C$ , how do we place it within the  $EC/EC_{\Delta}$  hierarchy?

- To show  $C$  is  $EC$ : Define a single sentence (or finite set of sentences)  $\varphi$  and show  $C = models(\varphi)$ .
- To show  $C$  is  $EC_{\Delta}$ : Define a set of sentences  $\Phi$  and show  $C = models(\Phi)$ .
- To show  $C$  is not  $EC$ : Use compactness.
- To show  $C$  is not  $EC_{\Delta}$ : Show that  $\overline{C}$  is  $EC_{\Delta}$  and  $\overline{C}$  is not  $EC$ .

Note that this is not an exhaustive approach. For example, if both  $C$  and  $\overline{C}$  are not  $EC_{\Delta}$ , then this method will not work. The class of connected graphs is an example of such a class. We will look at this example in the next lecture, when we discuss compactness.