Lecture 21: FOL Expressiveness - II

1 Review of Isomorphism

In the previous lecture we defined homomorphisms, which are mappings that preserve structure, and isomorphisms, which are bijective homomorphisms. We defined isomorphic equivalence, and proved the following important theorem:

Theorem 1 (Isomorphism Theorem)**.** *Let A and B be structures such that $A \simeq B$. Let φ be a sentence. Then $A \models \varphi$ iff $B \models \varphi$.*

So if two structures are isomorphic, then they satisfy the same sentences. The property that two structures satisfy the same sentences is interesting in itself. Recall that two sentences are defined to be logically equivalent if they are satisfied by the same structures. Similarly, we can naturally define a notion of equivalence for structures in terms of the formulas they satisfy. We call this *elementary equivalence.*

2 Elementary Equivalence

Definition 1 (Elementary Equivalence)**.** *Let A and B be structures on some vocabulary. We say that A and B are* elementarily equivalent*, denoted $A \equiv B$, if for all sentences φ, we have $A \models \varphi$ iff $B \models \varphi$.*

So two structures are elementary equivalent if they satisfy the same set of sentences. From the isomorphism theorem, we immediately get that if two structures are isomorphic then they are also elementary equivalent. Thus, isomorphism is a stronger notion of equivalence than elementary equivalence.

Theorem 2. *If $A \simeq B$ then $A \equiv B$.*

Example 1. $(\mathbb{N}, <) \not\equiv (\mathbb{R}, <)$. There is a smallest natural number but there is no smallest real number. Let $\varphi = \exists x \forall y (x \not\approx y \to x < y)$. Then $(\mathbb{N}, <) \models \varphi$ and $(\mathbb{R}, <) \not\models \varphi$.

Note that, as the example below shows, elementary equivalence does not imply isomorphic equivalence.

Example 2. $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ ($\mathbb{Q}$ is the set of rational numbers). This is because rational numbers and real numbers have the same order structure. For example, in both structures there is no smallest or greatest number, and between any two numbers there is another number. However, Q is countable and R is uncountable, so there can be no bijection between them. Thus $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$.

3 Definability

We now consider expressiveness of first order logic in terms of the mathematical objects it can define. For example, we can define the unit circle as the binary relation $\{(x, y) : x^2 + y^2 = 1\}$ on $\mathbb{R}$. We can also define the symmetry property for a binary relation R as $\forall x \forall y (xRy \leftrightarrow yRx)$, which is satisfied by all symmetric binary relations including the circle relation. Thus we can define particular instances of mathematical objects as well as general properties of these objects. This leads naturally to two distinct concepts of definability:

- Definability within a fixed structure.
- Definability of a class of structures.

We look at these two concepts in turn.

3.1 Definability within a fixed structure

Definition 2. *Given a structure $A = (D, P^A, \ldots, f^A, \ldots)$, we say that k-ary relation $R \subseteq D^k$ is* definable *in A if there exists a formula φ with $|FVARS(\varphi)| = k$, such that for all $(a_1, \ldots, a_k) \in D^k$,*

$$(a_1, \ldots, a_k) \in R \textit{ iff } A, [a_1, \ldots, a_k] \models \varphi$$

This notion of definability is a generalization of the concept of database queries discussed earlier in the course. If we no longer restrict the database to a finite domain and allow functions in addition to relations in the structure, then we can view the formula that defines a relation as a query over a (potentially infinite) database.

Note that restricting definability to relations still lets us define subsets, single elements and functions. We can define subsets of the domain as unary relations, and elements of the domain as singleton subsets. In order to define a unary function $f : D \to D$ we can define the relation $\{(a, f(a)) : a \in D\}$.

Example 3. We attempt to define 0 in different numerical structures. We see that we have to use different properties of 0 in different structures.

- $(N, <)$. Here we have access to the order relation of natural numbers and 0 can be defined as the smallest natural number. We define $\{0\}$ by $\forall y (y \not\approx x \to x < y)$
- $(\mathbb{R}, <)$. 0 has no special property with respect to the order relation, so it cannot be defined in this structure.
- $(\mathbb{R}, +)$. 0 is the additive identity and its own additive inverse. We can define $\{0\}$ in two ways: $\forall y (x + y \approx y)$ or $(x + x \approx x)$.

- $(\mathbb{R}, \cdot)$. 0 multiplied by any number is always zero. We define $\{0\}$ by $\forall y(x \cdot y \approx x)$. Note that, unlike the previous case, $(x \cdot x \approx x)$ is satisfied by both 0 and 1, so cannot be used to define 0.

Example 4. Given a graph $G = (V, E)$ as a mathematical structure, we would like to define the reachability relation:

$$Reach = \{(x, y) \in V^2 : \text{ there is a path in } G \text{ from } x \text{ to } y\}$$

We can define $Reach_0(x, y) = E(x, y)$ and for all $n \in N$,

$$Reach_{n+1}(x, y) = \exists z(Reach_n(x, z) \wedge E(z, y))$$

If V is finite, then any two connected vertices are connected by a path of length at most $|V|$. Then the formula $\bigvee_{i \leq |V|} Reach_i(x, y)$ defines the relation $Reach$. Thus reachability can be defined for finite graphs in first order logic. However, if V is not finite, then, as we will show later, reachability cannot be defined in first order logic.

3.2 Definability of a class of structures

Given a sentence φ, we define $models(\varphi) = \{A : A \models \varphi\}$. Thus $models(\varphi)$ is the *class* of all structures that satisfy φ. Why do we say class instead of set here? Consider $\varphi = \forall x(x \approx x)$. Then every set is a model of φ and $models(\varphi)$ contain all sets. Requiring $models(\varphi)$ to be a set then leads directly to Russell's paradox involving the set of all sets (Consider the set of all sets that are not members of themselves. Is this set a member of itself?). So, in general, $models(\varphi)$ is not a set but a class.

The question we would like to answer is: what classes of structures can we describe using first-order sentences? For example, can we describe the class of all connected graphs? Before we try to answer these questions, we make an important distinction between classes that can be expressed using a single formula, and classes that can be expressed using infinitely many formulas.

Definition 3 (EC)**.** *A class C of structures is* elementary*, denoted EC, if $C = models(\varphi)$ for some sentence φ.*

Examples: the class of undirected graphs, the class of groups.

Definition 4 (EC_Δ)**.** *A class C of structures is* extended elementary*, denoted EC_Δ, if $C = models(\Phi)$ for some set of sentences Φ.*

Examples: the class of infinite structures, the class of 2-colorable graphs.