Barycentric Coordinates for Convex Polytopes

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Abstract

An extension of the standard barycentric coordinate functions for simplices to arbitrary convex polytopes is described. The key to this extension is the construction, for a given convex polytope, of a unique polynomial associated with that polytope. This polynomial, the *adjoint* of the polytope, generalizes a previous two-dimensional construction described by Wachspress. The barycentric coordinate functions for the polytope are rational combinations of adjoints of various dual cones associated with the polytope.

1 Introduction

Let $v(S)$ denote the set of vertices of a simplex S. Functions $\{B_v(x)|v \in v(S)\}$ are *barycentric coordinates* with respect to S if the following three properties are satisfied.

- Non-negativity - For all v in $v(S)$,

$$B_v(x) \geq 0 \quad \forall x \in S.$$

- Linear precision - Given a linear function $L(x)$,

$$L(x) = \sum_{v \in v(S)} L(v) B_v(x).$$

- Minimal degree - For all v in $v(S)$, $B_v(x)$ is a linear polynomial.

If $L(x)$ is the constant one, then linear precision implies that the barycentric coordinates sum to one; they form a partition of unity.

Barycentric coordinates for a simplex are typically expressed as the ratios of volumes of various subsimplices. Figure 1 depicts a point p inside a triangle $\triangle v_1 v_2 v_3$. The barycentric coordinate of the point p with respect to v_3 is

$$\frac{area(\triangle v_1 v_2 p)}{area(\triangle v_1 v_2 v_3)}.$$

The areas of these triangles are linear functions of the position of p. Barycentric coordinates for simplices arise in a variety of applications. For example, linear interpolation is simple using barycentric coordinates: given $L(v)$ at the vertices of a simplex, the linear precision property can be used to compute $L(x)$. For this reason, barycentric coordinates for simplices and various extensions to other types of polytopes appear in the finite element literature [Wac75].

The Bernstein-Bézier representation of polynomial functions [Far88] also relies on barycentric coordinates. Given a simplex, barycentric coordinates span the space of linear functions over that simplex. Higher degree basis functions over that simplex can be constructed by applying the multinomial theorem to the sum of the basis functions. Linear precision is essential for guaranteeing that the space of all basis functions of degree k spans the space of all polynomials of degree k.

Barycentric coordinates also exist for more general types of polytopes. For example, the tensor product of the barycentric coordinate functions for a set of simplices are barycentric over the tensor product of simplices. The unit square is the tensor of two unit intervals. Barycentric coordinates over the square are exactly the bi-linear basis functions.

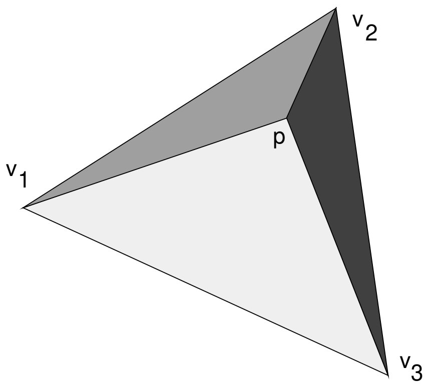

Figure 1 Barycentric coordinates for a triangle

The goal of this paper is to construct rational coordinate functions for arbitrary convex polytopes that are nonnegative, have linear precision, and have minimal degree. The degree of these rational functions depends on the number of facets of the polytope and its dimension. Several others have investigated this problem in the case of convex polygons. Wachspress [Wac75] defines barycentric finite element basis functions for convex m-gons. These functions are rational and of degree $m - 2$. Wachspress also generalizes this construction to special types of polyhedra. Loop and DeRose [LD89] independently develop similar functions for defining multi-sided surface patches used in Computer-Aided Geometric Design. In the case of general convex polytopes, Lee [Lee89] has developed non-rational basis functions that are non-negative and have linear precision. In this paper, we focus on building rational basis functions that are barycentric for arbitrary convex polytopes.

The next section reviews some of the properties of convex polytopes and polyhedral cones in particular. Subsequent sections use these ideas to define a fundamental polynomial associated with a polyhedral cone. This polynomial, the *adjoint* of the cone, is the key building block in creating barycentric coordinate functions.

2 Polyhedral cones

A convex *polytope* P is the convex hull of a non-empty, finite set of points. If the set of points contains $d + 1$ affinely independent points, then P has dimension d. A *d-simplex* is the convex hull of $d + 1$ affinely independent points. A *face* of P is the intersection of P and a supporting hyperplane for P. If P has dimension d, then the *facets* of P are the $(d - 1)$-dimensional faces of P. The *edges* and *vertices* of P are faces of dimension one and zero, respectively. Grunbaum [Gru67] contains a complete exposition of the theory of convex polytopes.

The convex hull of a set of rays emanating from the origin is a *polyhedral cone* (with apex at the origin). One natural way to create a polyhedral cone is to embed a given polytope P in a hyperplane ($x_0 = 1$) and take the set of all rays from the origin through P. The intersection of this polyhedral cone C in R^{d+1} with the hyperplane ($x_0 = 1$) is exactly P. (P is the *cross section* polytope for C.)

The dimension of C, $dim(C)$, is simply the dimension of P. (Note that this definition is nonstandard, but simplifies subsequent dimension formulas). Vertices of C, $v(C)$, correspond to those rays that intersect vertices of P. For the sake of uniqueness, we represent this ray by a unit length vector v. The facets of C, $f(C)$, correspond to the facets of P. If C has maximal dimension, then each facet F has a unique inward unit normal n_F.

One reason for our use of polyhedral cones is that they are projective; parallel faces cannot exist. Our second reason is that duality plays a large role in constructing barycentric coordinates and duality is more easily defined in terms of cones. Given a polyhedral cone C, its dual cone $D[C]$ is the convex hull of those rays r such that C lies in the positive halfspace defined by r, $\{x|r \cdot x \geq 0\}$.

Duality induces an isomorphism between faces of C and faces of $D[C]$. If v is a vertex of C, then $D[v]$ is the halfspace bounding a facet of $D[C]$. If F is a facet of C, then $D[F]$ is a cone defined by those halfspaces bounding a vertex of $D[C]$. More generally, if F is an i-dimensional face of C, then $D[F]$ is a cone defined by those halfspaces bounding a $(d-i-1)$-dimensional face of $D[C]$. Note that this isomorphism obeys the classical characterization of duality: face containment is reversed. If $F_1 \subseteq F_2$, then $D[F_2] \subseteq D[F_1]$.

3 Adjoints of polyhedral cones

Wachspress defines the adjoint of a polygon P as the minimal degree algebraic curve interpolating the intersection points of pairs of lines containing non-adjacent edges of P. This approach has two drawbacks. First, the polynomials defining the algebraic curve are only unique up to multiplication by a constant. Defining barycentric coordinates requires a careful choice of this constant. Second, this interpolation problem is very difficult to pose correctly in higher dimensions.

Our approach is to associate a unique polynomial $A[C](x)$ with a polyhedral cone C. If C is the cone associated with the original polytope P, then the zero contour of $A[D[C]](x)$ is the adjoint in the sense of Wachspress. This approach avoids the normalization problem and works in complete generality. We later show that this construction is equivalent to Wachspress's construction in the case of convex polygons.

3.1 Triangulations of cones

If the vertex rays of a polyhedral cone S are linearly independent, then S is a *simplicial* cone. The cross-section polytope of a simplicial cone is a simplex. Just as a convex polytope can be triangulated using simplices, a polyhedral cone can be triangulated using simplicial cones.

Definition: A triangulation of a d-dimensional polyhedral cone C is a partition of C into a collection of d-dimensional simplicial cones, $t(C)$, such that

- The vertex rays of each simplicial cone are a subset of the vertex rays of C.
- The union of the simplicial cones is exactly C.
- The intersection of any two simplicial cones is a face of both.

The Delaunay triangulation of a convex polytope is a well-known example of a triangulation. [PS85] is good source for more material on triangulations.

3.2 A definition of adjoints

If S is a simplicial cone, then let a_S denote the volume of the parallelepiped spanned by the vertex rays in $v(S)$. These volume play an important role as normalizing constants in the definition of adjoints. If v is the vertex ray opposite a facet F of S, then a_S, v, and n_F are related in the following manner

$$a_S = (v \cdot n_F)a_F. \tag{1}$$

The volume of the parallelepiped is the volume of its base, a_F, time its height, the dot product of v with the unit normal n_F.

Definition: Given a polyhedral cone C and a triangulation $t(C)$, the adjoint of C is

$$A[C](x) = \sum_{S \in t(C)} a_S \prod_{v \in v(C)-v(S)} (v \cdot x).$$

In the case of a simplicial cone S, the adjoint $A[S](x)$ is simply the constant a_S.

As stated, the definition of the adjoint of C appears to depend on the particular triangulation chosen. Using theorem 4, the adjoint is shown to be independent of the particular triangulation. Therefore, we do not index the adjoint by the particular triangulation.

Theorem 1 *$A[C](x)$ is a homogeneous polynomial of degree $|v(C)| - dim(C) - 1$ that is nonnegative for all $x \in D[C]$.*

Proof: By definition, a_S is nonnegative. Similarly, $(v \cdot x)$ is also nonnegative on C for all $v \in v(C)$. Therefore, $A[C](x)$ must be nonnegative for all $x \in D[C]$. Since $|v(S)| = dim(C)+1$, the number of the homogeneous terms in the definition is exactly $|v(C)| - dim(C) - 1$. $\square$

4 Properties of adjoints

4.1 A recurrence for adjoints of simplicial cones

The next theorem relates the adjoint of a simplex to the adjoints of its facets. This relation is the starting point for constructing coordinate functions with linear precision.

Theorem 2 *If S has maximal dimension, then for any linear function $L(x)$*

$$L(x)A[S](x) = \sum_{F \in f(S)} L(n_F)A[F](x)(v \cdot x). \tag{2}$$

where v is the vertex opposite the facet F in S.

Proof: If equation 2 holds when $L(x) = (w \cdot x)$ for all $w \in v(S)$, then the theorem must hold for any linear $L(x)$ since S has maximal dimension. It remains to show that

$$(w \cdot x)a_S = \sum_{F \in f(S)} (w \cdot n_F)a_F(v \cdot x). \tag{3}$$

where v is the vertex opposite the facet F in S. If $w \neq v$, then $w \in v(F)$ and $w \cdot n_F = 0$. When $w = v$, equation 3 reduces to

$$(w \cdot x)a_S = (v \cdot n_F)a_F(w \cdot x).$$

The theorem follows by equation 1. $\square$

If the simplex S does not have maximal dimension, then the linear function $L(x)$ must be restricted to be in the span of the functions $(v \cdot x)$ where $v \in v(S)$. A similar proof holds as long as the normal n_F is now chosen to lie in the span of S.

4.2 A recurrence for adjoints of polyhedral cones

The next theorem establishes a relation similar to that of equation 2 for polyhedral cones. At a conceptual level, its proof is similar to that of Stokes' theorem [Fle77] in which the integral over an object is related to an integral over the boundary of the object. In this case, the object is the cone C. The adjoint of C is defined as a sum involving the adjoints of simplicial cones triangulating C. Equation 2 relates the adjoint of a simplicial cone to the adjoints of its facets. Composing these two equations, the adjoint of C is a sum over the adjoints of facets of the triangulation of C.

Terms corresponding to interior facets appear twice in the sum and cancel out. The remaining facets tile the boundary of C. These exterior facets form a triangulation of facets of C. Thus, the adjoint of a polyhedral cone is related to the adjoints of its facets.

Theorem 3 *Let C have maximal dimension, then for any linear function $L(x)$,*

$$L(x)A[C](x) = \sum_{F \in f(C)} L(n_F)A[F](x) \prod_{v \in v(C)-v(F)} (v \cdot x). \tag{4}$$

subject to $t(C)$ coinciding with $t(F)$ on each facet F.

Proof: Recalling the definition of $A[C](x)$ and multiplying both sides by $L(x)$ yields

$$L(x)A[C](x) = \sum_{S \in t(C)]} L(x)\ a_S \prod_{v \in v(C)-v(S)} (v \cdot x).$$

Replacing $L(x)a_S$ by the right hand side of equation 2 yields

$$L(x)A[C](x) = \sum_{S \in t(C)} (\sum_{F \in f(S)} L(n_F)A[F](x) \prod_{v \in v(C)-v(F)} (v \cdot x)). \tag{5}$$

If F is an interior facet of $t(C)$, then F is shared by two distinct simplicial cones in $t(C)$ on opposite sides of F. Since n_F is oriented to point inward with respect to each of the these simplicial cones, the two terms corresponding to an interior facet agree except for sign and must cancel out. The remaining facets on the exterior of C can be organized in the following fashion.

If S is a simplicial facet of $t(C)$ lying on a facet F of C, then the set of all such S form a triangulation, $t(F)$, of F (as stated in the theorem). Thus, equation 5 can be rewritten as

$$L(x)A[C](x) = \sum_{F \in f(C)} (\sum_{S \in t(F)} L(n_S)A[S](x) \prod_{v \in v(C)-v(S)} (v \cdot x)).$$

Since $n_S = n_F$ for all $S \in t(F)$, the term $L(n_S)$ can be replaced by $L(n_F)$ and moved outside the inner sum. Finally, the final product term can now split into two parts.

$$L(x)A[C](x) = \sum_{F \in f(C)} L(n_F)(\sum_{S \in t(F)} A[S](x) \prod_{v \in v(F)-v(S)} (v \cdot x)) \prod_{v \in v(C)-v(F)} (v \cdot x).$$

By definition, the inner sum is the adjoint of $A[F](x)$. Therefore,

$$L(x)A[C](x) = \sum_{F \in f(C)} L(n_F)A[F](x) \prod_{v \in v(C)-v(F)} (v \cdot x).$$

$\square$

If the cone C does not have maximal dimension, then the linear function $L(x)$ must be restricted to lie in the span of the functions $(v \cdot x)$ where $v \in v(C)$. Using a reduced dimension version of theorem 2 establishes a similar theorem. By relating the adjoint of a cone C to the adjoints of its facets, we can prove the uniqueness of the adjoint.

Theorem 4 *$A[C](x)$ is independent of the triangulation $t(C)$.*

Proof: Proceed by induction on the dimension of C. If the dimension of C is zero, then C is a ray and poses a single triangulation. If F is a facet of C, then the dimension of F is less than the dimension of C. So, by the inductive hypothesis, $A[F](x)$ is triangulation independent. Therefore, applying equation 4, $A[C](x)$ must be independent of the particular triangulation $t(C)$. $\square$

4.3 Planar restrictions of adjoints

The adjoint $A[C](x)$ vanishes on certain important linear subspaces. This property is necessary to ensure that barycentric coordinate functions have the proper degree when restricted to the faces of a polytope. Specifically, if the two vertex rays v and w are not edge adjacent in C, then $A[C](x)$ vanishes on the linear space orthogonal to v and w.

Theorem 5 *Let v and w be a pair of vertices of C that are not edge adjacent. Then $A[C](x) \equiv 0$ on the subspace*

$$\{x | (v \cdot x = 0) \cap (w \cdot x = 0)\}.$$

Proof: The proof proceeds by induction on the dimension of C. If C is a single ray, then C has no edges and the theorem follows. Otherwise, equation 4 holds

$$L(x)A[C](x) = \sum_{F \in f(C)} L(n_F) A[F](x) \prod_{v \in v(C) - v(F)} (v \cdot x).$$

We next show that each term in the righthand sum vanishes on the appropriate space. If either of v or w is not in $v(F)$, then the product term contains one of $(v \cdot x)$ or $(w \cdot x)$. Otherwise, both v and w are in $v(F)$. Since v and w are not edge adjacent in C, they are also not edge adjacent in F. By the inductive hypothesis, $A[F](x)$ must vanish on $(v \cdot x = 0) \cap (w \cdot x = 0)$. $\square$

5 Barycentric coordinates

Berycentric coordinates for polyhedral cones can be characterized by the same three properties used for simplices. If C is a polyhedral cone, then the functions $\{B_v(x)|v \in v(C)\}$ are *homogeneous barycentric coordinates* with respect to C if the following three properties are satisfied.

- Non-negativity - For all v in $v(C)$,

$$B_v(x) \geq 0 \quad \forall x \in C.$$

- Linear precision - Given a linear function $L(x)$,

$$L(x) = \sum_{v \in v(C)} L(v) B_v(x).$$

- Minimal degree - For all v in $v(C)$, $B_v(x)$ is a homogeneous rational function of degree $|f(C)| - dim(C)$.

The first two properties mirror the simplex definition. The third property, minimal degree, guarantees that the barycentric coordinate functions have the lowest degree possible and are unique. A proof of this fact will appear in a subsequent paper.

5.1 Barycentric coordinates for polyhedral cones

Given a polyhedral cone C, let D be the cone dual to C. Then, the barycentric coordinate functions for C can be derived directly from equation 4.

$$L(x)A[D](x) = \sum_{F \in f(D)} L(n_F) A[F](x) \prod_{v \in v(D)-v(F)} (v \cdot x).$$

Dividing both sides by $A[D](x)$ yields that

$$L(x) = \sum_{F \in f(D)} L(n_F) \frac{A[F](x) \prod_{v \in v(D)-v(F)} (v \cdot x)}{A[D](x)}.$$

We next reinterpret this equation in terms of C instead of D. Vertex rays of D are dual to halfspaces defining facets in C. Facets of D are dual to certain cones associated with the vertices of C. If $c(v)$ denotes the intersection of those halfspaces bounding v, then F is dual to $c(v)$. Under this interpretation,

$$L(x) = \sum_{v \in v(C)} L(v) \frac{b_v(x)}{d(x)}, \tag{6}$$

where

$$\begin{aligned} b_v(x) &= A[D[c(v)]](x) \prod_{F \in f(C), v \notin v(F)} (n_F \cdot x), \\ d(x) &= A[D[C]](x). \end{aligned}$$

For a vertex v, $b_v(x)$ is the product of the adjoint of the dual to the cone of v and the equations of faces opposite v. Note that the functions $\frac{b_v(x)}{d(x)}$ have linear precision and are nonnegative by construction. The polynomial $d(x)$ has degree

$$|v(D)| - dim(D) - 1 = |f(C)| - dim(C) - 1$$

by theorem 1. The degree of $b_v(x)$ is

$$|v(F)| - dim(F) - 1 + |v(D) - v(F)| = |v(D)| - dim(D) = |f(C)| - dim(C).$$

Thus, these rational functions are the homogeneous barycentric coordinates.

5.2 Barycentric coordinates on facets

Perhaps the most pleasing aspect of this construction is that the resulting barycentric coordinate behave correctly on facets of C. Since the basis functions are nonnegative and have linear precision over C, they also inherit these properties over any facet F of C. A more interesting question concerns the degree of these basis function when restricted to F. For example, if F is a simplex, the barycentric coordinates should simply be linear functions on F. In fact, the degree of these basis functions on F is minimal as shown in the next theorem.

Theorem 6 *If F is a facet of C, then the restriction of $\frac{b_w(x)}{d(x)}$ to $(n_F \cdot x = 0)$ is a rational function of degree $|f(F)| - dim(F)$.*

Proof: The remaining facets of C (excluding F) can be characterized as one of two types.

- Facets whose intersection with F defines a facet of F,

- Facets whose intersection with F is either empty or a lower dimensional face of F.

Let v be the vertex of D dual to F. If $\hat{F}$ is a facet in the second category, then its dual $\hat{v}$ on D is not edge adjacent to v. By theorem 5, the adjoint $A[D](x)$ (and the terms comprising the righthand side of equation 4) vanishes on $(v \cdot x = 0) \cap (\hat{v} \cdot x = 0)$. So, on $(v \cdot x = 0)$, the functions $b_w(x)$ and $d(x)$ share a common linear factor.

The degree of $\frac{b_w(x)}{d(x)}$ on C is $|f(C)| - dim(C) = |f(C)| - dim(F) - 1$ by theorem 1. Since the degree of this function decreases by one for each of the $|f(C)| - 1 - |f(F)|$ facets in the second category, the degree of $\frac{b_w(x)}{d(x)}$ restricted to $(v \cdot x = 0)$ is exactly $|f(F)| - dim(F)$. □

Wachspress [Wac75] first introduced the notion of adjoints in his construction of finite element bases for convex m-gons. Given a convex m-gon P, the m lines bounding P intersect in exactly $\frac{(m)(m-1)}{2}$ points (including points at infinity); m of these points are vertices of P. According to Wachspress, the adjoint of P is the unique algebraic curve of degree $m - 3$ that interpolates the remaining $\frac{(m)(m-3)}{2}$ points where the extensions of non-adjacent edges intersect. Figure 2 illustrates the situation in the case of a pentagon.

If C is the cone associated with P, then vertices of $D[C]$ correspond to edges of C (and P). Two edges of C share a vertex if and only if their dual vertices in $D[C]$ are adjacent. By theorem 1, the adjoint of $D[C]$ vanishes at the intersection of those edges of C that do not intersect in a vertex. Since the adjoint has degree $m - 3$, its zero set must be the adjoint

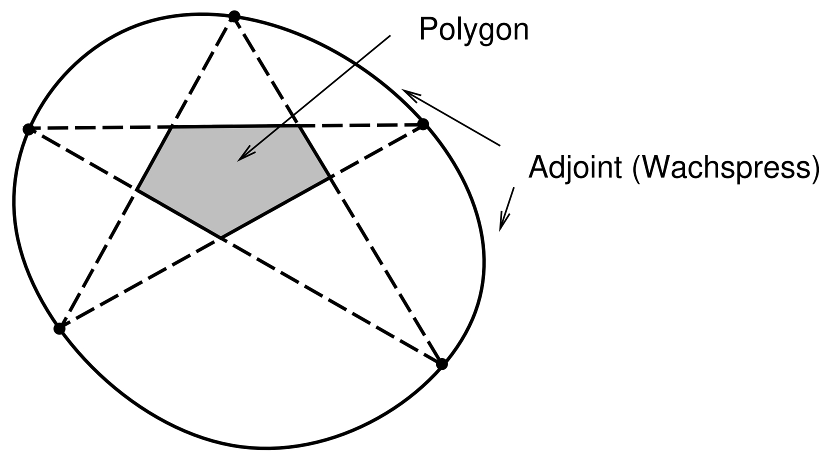

Figure 2 A pentagon and its adjoint curve

in the sense of Wachspress.

5.3 Barycentric coordinates for convex polytopes

The restriction of the cone C to $(x_0 = 1)$ is the original convex polytope P. If $v = (v_0, v_1, ..., v_d)$ is a vertex ray of C, then $(1, \frac{v_1}{v_0}, ..., \frac{v_d}{v_0})$ is the corresponding vertex of P. Using this observation, we can view rewrite equation 6 to evaluate L at the vertices of P.

$$L(x) = \sum_{v \in v(C)} L(1, \frac{v_1}{v_0}, ..., \frac{v_d}{v_0}) \frac{v_0 b_v(x)}{d(x)}.$$

Dehomogenizing this equation yields the affine barycentric basis functions for P. Since P lies in $(x_0 = 1)$, v_0 must be positive and the basis functions remain nonnegative.

6 Future work

Several areas associated with this problem merit further research. Barycentric coordinates for simplices can be defined solely in terms of volumes of various simplices. Loop and DeRose construct barycentric coordinates for convex polygons in terms of the areas of various triangles. A similar construction for polytopes would yield a more natural definition of barycentric coordinates without resorting to the homogeneous formulation of polyhedral cones.

Another interesting question is whether barycentric coordinates as defined here are unique. The answer is yes. A proof of this fact will appear in a subsequent paper. One consequence is that the barycentric coordinates described here are equivalent to coordinate functions created by tensoring barycentric coordinate functions for simplices.

The proof of theorem 3 establishes that the adjoint of C can be viewed as the sum of various terms associated with triangulations of facets of C. This observation can be used to derive a fast method for evaluating adjoints. For tensors of simplices, this method is competitive with standard multi-linear interpolation. This problem will also be addressed in a subsequent paper.

Finally, applications of barycentric coordinates for polyhedra merit exploration. Such functions could be used to create continuous trivariate maps over a partition of space into convex polyhedra. Free-form deformations over such partitions would now be straightforward. The center of figure 3 depicts a sphere embedded in octahedron. The sphere is represented in terms of barycentric coordinates over the original octahedron. The left and right objects are deformations of the sphere as the defining octahedron is deformed.

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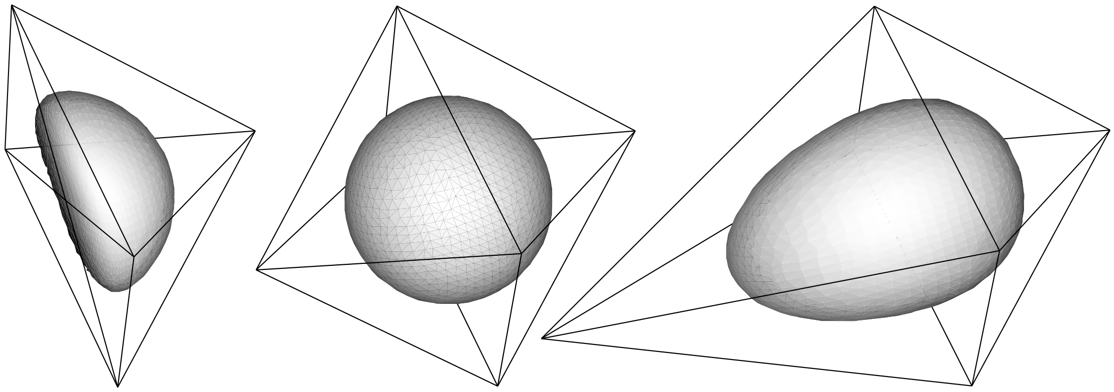

Figure 3 An sphere embedded in an octahedron and its deformations

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